

Nonsymmetric entropy I: basic concepts and results

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Abstract

A new concept named nonsymmetric entropy which generalizes the concepts of Boltzman's entropy and shannon's entropy, was introduced. Maximal nonsymmetric entropy principle was proven. Some important distribution laws were derived naturally from maximal nonsymmetric entropy principle.

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1 introduction

Entropy which measures the uncertain degree of information is an important basic concept in statistic physics and information theory. In Ref.[1] and its references, entropy is discussed from many aspects. In present paper, I give a new entropy named nonsymmetric entropy which measures the average value of the auxiliary and probability two kinds of information to an event. I prove the corresponding maximal nonsymmetric entropy principle. Some interesting distribution laws can be derived naturally from this principle.

2 Basic conceptions and results

Firstly we give some concepts in the following.

Definition 1:The auxiliary information of an event x_i is defined by

$$A(x_i) = -\ln(\beta_i), \quad (1)$$

where β_i is auxiliary information parameter.

Definition 2: Total information of an event x_i is

$$It(x_i) = I(x_i) + A(x_i) = -\ln p(i) - \ln \beta_i = -\ln(\beta_i p(i)). \quad (2)$$

Definition 3: we define a function

$$S_m(p(1), \dots, p(m)) = -\sum_{i=1}^m p(i) \ln(\beta_i p(i)), \quad (3)$$

where $\beta_i > 0, i = 1, \dots, m$, are nonsymmetric parameters. We call the function S_m the nonsymmetric entropy.

Remark 1: if we take $\beta_i \equiv 1$, we get the Shannon's entropy, thus nonsymmetric entropy generalizes the concept of Shannon's entropy.

It is obvious that nonsymmetric entropy measures the expect information of the total information of all events. Because we consider auxiliary information so that we can describe event in a more right way. At the same time, we can use nonsymmetric parameters β_i to derive some important distribution laws which include Zipf's law. In particular, if we take

$$\beta_i = i^\alpha, \quad (4)$$

then the corresponding nonsymmetric entropy becomes

$$S = -\sum_{i=1}^m p(i) \ln(i^\alpha p(i)), \quad (5)$$

we can call it Zipf's entropy. We have the following result:

Theorem 1: If $p(i)$ satisfies the following Zipf's distribution law

$$p(i) = \frac{p(1)}{i^\alpha}, \quad (6)$$

then the Zipf's entropy takes the maximum.

Corollary 1. For Zipf's law, we have $S_{m+1} > S_m$, that is, the nonsymmetric entropy is increase as m increasing.

From the above theorem it is easy to see that the Zipf's law can be derived naturally from nonsymmetric entropy under some special auxiliary parameters. We don't prove this theorem at the present time, in fact we have the following more general result.

Theorem 2: If $\{p(1), \dots, p(m)\}$ satisfies the following distribution

$$p(i) = \frac{1}{\beta_i \sum_{i=1}^m \frac{1}{\beta_i}}, \quad (7)$$

then the nonsymmetric entropy S_m takes the maximum

$$S_m = -\ln p(1) = \ln \sum_{i=1}^m \frac{1}{\beta_i}. \quad (8)$$

Proof: Instituting $p(m) = 1 - p(1) - \dots - p(m-1)$ into the Eq.(3) and setting its differential to zero yields

$$\frac{\partial S_m}{\partial p(i)} = -\ln \frac{\beta_i p(i)}{\beta_m (1 - p(1) - \dots - p(m-1))} = 0, \quad i = 1, \dots, m-1, \quad (9)$$

that is,

$$p(1) + \dots + (1 + \frac{\beta_i}{\beta_m})p(i) + \dots + p(m-1) = 1, \quad i = 1, \dots, m-1. \quad (10)$$

Solving the equations system (10), we obtain

$$p(i) = \frac{1}{\beta_i \sum_{j=1}^m \frac{1}{\beta_j}} = \frac{\beta_1}{\beta_i} p(1), \quad i = 1, \dots, m, \quad (11)$$

where $p(1) = \frac{1}{\beta_1 \sum_{j=1}^m \frac{1}{\beta_j}}$. Denote $A_k = (a_{ij})_{k \times k}$, $a_{ij} = \frac{\delta_{ij}}{p(i)} + \frac{1}{p(m)}$, since $\det A_k = \frac{1}{p(m)} \sum_{i=1}^k \frac{1}{p(1) \dots \hat{p(i)} \dots p(k)}$, where the hat means the corresponding item disappear, so from $a_{ij} = -\frac{\partial^2 S_m}{\partial p(i) \partial p(j)} = \frac{\delta_{ij}}{p(i)} + \frac{1}{p(m)}$, we know that the matrix $A = (a_{ij})_{(m-1) \times (m-1)}$ is a positive defined matrix. Thus the distribution $p(i) = \frac{1}{\beta_i \sum_{i=1}^m \frac{1}{\beta_i}}$ maximize the nonsymmetric entropy. The proof is completed.

Remark 2: If we take $\beta_i = i^\alpha$, then we have

$$p(i) = \frac{1}{\sum_{j=1}^m \frac{1}{j^\alpha}} = \frac{p(1)}{i^\alpha}, \quad i = 1, \dots, m, \quad (12)$$

in particular, we take $\alpha \simeq 1$, this is just the Zipf's law in linguist. If take $\beta_i = (i + \gamma)^\alpha$, then we give Mandelbrot's law. If we take other values of β_i , we will give other distribution law. Thus the key is to choose suitable auxiliary information parameters β_i , this is a problem need to study deeply.

Remark 3: Using maximal nonsymmetric entropy principle in Section 3, we can give a simple proof for theorem 2.

We consider the continuous case in the following.

Definition 4: For continuous case, nonsymmetric entropy is defined

$$S(\rho) = - \int \rho(x) \ln\{\beta(x)\rho(x)\} dx, \quad (13)$$

where $\beta(x)$ is auxiliary information parameter function, $\rho(x)$ is probability density of event.

In order to solve maximal nonsymmetric entropy distribution, we can use variant method. Under some constrains conditions, we use lagrange multiply

method to do this thing. We give several example in the following to illustrate our method.

Theorem 2: Assume $\int x\rho(x)dx = \mu$, we then its maximal nonsymmetric entropy distribution is

$$\rho_0(x) = \frac{1}{\beta(x)} \exp(1 - \lambda_1 - \lambda_2 x), \quad (14)$$

where λ_1 and λ_2 satisfy two constrain conditions $\int \frac{1}{\beta(x)} \exp(1 - \lambda_1 - \lambda_2 x) dx = 1$ and $\int \frac{x}{\beta(x)} \exp(1 - \lambda_1 - \lambda_2 x) dx = \mu$.

Proof: Make a auxiliary functional

$$F(\rho, \lambda_1, \lambda_2) = - \int \rho(x) \ln\{\beta(x)\rho(x)\} dx + \lambda_1 \left(\int \rho(x) dx - 1 \right) + \lambda_2 \left(\int x\rho(x) dx - \mu \right). \quad (15)$$

We have

$$\delta F = \int \{\lambda_1 + \lambda_2 x - \ln(\beta(x)\rho(x)) - 1\} \delta \rho(x) dx, \quad (16)$$

form $\delta F = 0$, we solve out as follows:

$$\rho_0(x) = \frac{1}{\beta(x)} \exp(1 - \lambda_1 - \lambda_2 x), \quad (17)$$

where λ_1 and λ_2 satisfy two constrain conditions $\int \rho_0(x) dx = 1$ and $\int x\rho_0(x) dx = \mu$.

Theorem 3: Assume $\int x\rho(x)dx = \mu$ and $\int x^2\rho(x)dx = \sigma^2$, we then its maximal nonsymmetric entropy distribution is

$$\rho_0(x) = \frac{1}{\beta(x)} \exp(1 - \lambda_1 - \lambda_2 x - \lambda_3 x^2), \quad (18)$$

where λ_1, λ_2 and λ_3 satisfy three constrain conditions $\int \rho_0(x) dx = 1$, $\int x\rho_0(x) dx = \mu$ and $\int x^2\rho_0(x) dx = \sigma^2$.

Proof: it is similar with the proof of theorem 2.

3 Maximal nonsymmetric entropy principle

We generalize the maximal entropy principle in information theory to the case of nonsymmetric entropy.

Definition 5: Denote Λ be a class of probability density functions, if $\rho_0 \in \Lambda$, such that

$$S(\rho_0) = \max\{S(\rho) : \rho \in \Lambda\}, \quad (19)$$

then ρ_0 is called maximal nonsymmetric entropy distribution, and $S(\rho_0)$ maximal nonsymmetric entropy.

Theorem 4 (maximal nonsymmetric entropy principle): Λ is a fixed class of probability density functions, if there exists $\rho_0 \in \Lambda$ such that

$$- \int \rho(x) \ln\{\beta(x)\rho_0(x)\} dx = S_0 \quad (20)$$

is a constant which is irrelative to $\rho(x)$ for every ρ , then $\rho_0(x)$ is maximal nonsymmetric entropy distribution, and $S(\rho_0) = S_0$ is maximal nonsymmetric entropy. For discrete case, this theorem is also right.

Proof: for arbitrary $\rho \in \Lambda$, we have

$$\begin{aligned} S(\rho) &= - \int \rho(x) \ln\{\beta(x)\rho(x)\} dx = - \int \rho(x) \ln\{\beta(x)\rho_0(x) \frac{\rho(x)}{\rho_0(x)}\} dx \\ &= - \int \rho(x) \ln\{\beta(x)\rho_0(x)\} dx - \int \rho(x) \ln \frac{\rho(x)}{\rho_0(x)} dx \\ &\leq - \int \rho(x) \ln\{\beta(x)\rho_0(x)\} dx = S_0, \end{aligned} \quad (21)$$

then S_0 is maximal nonsymmetric entropy. Since $S(\rho_0) = S_0$, so $\rho_0(x)$ is maximal nonsymmetric entropy distribution. The proof is completed.

Corollary 2: If $\int \frac{1}{\beta(x)} dx < \infty$, we have

$$\rho_0(x) = \frac{C}{\beta(x)} \quad (22)$$

is maximal nonsymmetric entropy distribution, where $C = \frac{1}{\int \frac{1}{\beta(x)} dx}$.

If we take $\beta(x) = x^\alpha, \alpha > 1$, and assume the arrange of random variable X is $(k, +\infty)$, then maximal nonsymmetric entropy distribution is

$$\rho_0(x) = \frac{k^{1-\alpha}}{\alpha-1} x^{1-\alpha}, \quad (23)$$

it is just the power law distribution in continuous case. If there are some constraints we will get other distributions similar with them in theorem 2 and theorem 3. On the other hand, we can easily use the maximal nonsymmetric entropy principle to give new proof s to theorems 2 and 3.

4 Discussions

The above results suggest that the nonsymmetric entropy is a rather fundamental concept that will play an important role in some fields. Perhaps the meaning of the nonsymmetric entropy needs a reasonable explanation. It is different with Shannon's entropy $S = -\sum_{i=1}^m p(i) \ln p(i)$ in some aspects. For example, if $p(j) = 1$ and others zeroes, then $S = 0$, but $S_m = -\ln \beta_j$, this implies that there exist some kind of uncertainty in some superficial reliable events under the nonsymmetric entropy. Other deep meanings of nonsymmetric entropy need more studies. Since the important roles of Boltzman's entropy and Shannon's entropy in thermodynamics and information theory respectively, we hope that maximal nonsymmetric entropy principle can play a suitable role in corresponding fields.

For example, Zipf's law ([2]) which states that the frequency of a word decays as a power law of its rank, is regarded as a basic hypothesis with no need for

explanation in recent models of the evolution of syntactic communication([3]). As an empirical law, Zipf's law is the most fundamental fact in quantitative linguistics, its meaning is still an open problem which has been tried to explain from several aspects of its origins([4, 5, 6, 7, 8]. It is necessary to find a suitable mechanism for Zipf's law. In this paper, Zipf's law is derived naturally by maximizing the nonsymmetric entropy when auxiliary parameter take some special values. It is at least need to consider seriously the meaning of those results. I will continue to study the theory and applications of nonsymmetric entropy.

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